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Multiple Criteria Decision Making: selecting variants along compromise lines[2]

Wielokryterialne Podejmowanie Decyzji: wybór wariantów wzdłuż linii kompromisu

Abstract
We consider multiple criteria decision making problems in which decision alternatives are evaluated against a given set of criteria. To solve problems of this class, whether interactively or in one step, a common approach is to approximate preferences articulated by the decision maker, iteractively or a priori, by *scalarizing functions*. Such functions when optimized over the set of feasible alternatives yield efficient alternatives.

The aim of this paper is to point to deficiencies of such an approach. Their roots are both in the inherent nature of some classes of decision problems, but also in the scalarizing function approach itself, at least as it is currently formulated and interpreted.

We present a multiple criteria decision making setting, which still makes use of scalarizing functions, but encapsulates DM's preferences and fulfils his expectations in, we believe, a more accurate way.

Streszczenie
Rozpatrywać będziemy problemy wielokryterialnego podejmowania decyzji, w których warianty decyzyjne są oceniane względem danego zestawu kryteriów. Dla rozwiązywania problemów tej klasy, zarówno w sposób interakcyjny jak i w jednym kroku, standardowym podejściem jest aproksymowanie preferencji wyrażanych przez decydenta, w trybie intrakcyjnym lub apriorycznie, za pomocą *funkcji skalaryzujących*. Optymalne wartości tych funkcji na zbiorze dopuszczalnych wariantów, wyznaczają warianty efektywne.

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Celem artykułu jest wskazanie wad takiego podejścia. Źródła tych wad tkwią zarówno we własnściach pewnych klas problemów decyzyjnych, jak również także w samej koncepcji funkcji skalaryzujących, co najmniej w postaci, w jakiej jest ona obecnie formułowana i interpretowana.

Przedstawimy pewien formalizm dla wielokryterialnego podejmowania decyzji, który ujmuje preferencje decydenta i spełnia jego oczekiwania w sposób, jak sądzimy, bardziej odpowiedni.

1. Introduction

Multiple Criteria Decision Making (MCDM) problems are solved either interactively via a sequence of evaluations of alternatives and preference expressing, or via a single-run of an algorithm, which derives a single alternative. When solving a decision problem in the interactive manner, the solution to the problem, the *most preferred alternative*, is the alternative which in the course of interactive decision process the DM evaluates as *the most preferred*. When solving a decision problem in the single-run manner, the solution to the problem, the most preferred alternative, is the alternative which satisfies best DM's preferences expressed *a priori*. An example of decision making problems which are to be solved in the latter manner are public procurement problems, where the alternative selection algorithm has to be specified and made public before tenders are made, and as a rule neither the algorithm none any of its parameters can be changed henceforth.

Candidates for the *most preferred alternative*, i.e. *efficient alternatives* are commonly derived by means of *scalarizing functions*. Given a scalarizing function, efficient alternatives are derived by optimization of this function over the set of feasible alternatives.

The aim of this paper is to point to deficiencies of the scalarizing function approach. Roots of such deficiencies are both in the inherent nature of some classes of decision problems but also in the scalarizing function approach itself, at least as it is currently formulated and interpreted. We present a decision making setting for problems considered, which encapsulates DM's preferences and fulfils his expectations in, we believe, a more accurate way. We argue that for the DM a natural and versatile way to express his preferences is to select a *consensus line*, i.e. a line which defines *proportions* between values of criteria, when they all increase or all decrease, he wants to be kept when searching for an efficient alternative.

The outline of the paper is as follows. In Section 2 we formulate multiple criteria decision making problems and introduce the necessary notation. In Section 3 we discuss methods of DM preference capturing based on scalarizing functions and we point to their deficiencies. In Section 4 we propose a preference capturing approach based on the notion of consensus line and we provide means for such an approach to be operational. An illustrative numerical example is presented in Section 5. In Section we discuss the possible use of the proposed approach in MCDM. Section concludes.

2. Problem Formulation

An MCDM problem is formalized as follows:

$$\text{choose an alternative } x \text{ for which vector } f(x),\ x \in X_0 \subseteq \mathcal{X}, \quad \text{is the most preferred,} \tag{1}$$

where $\mathcal{X}$ is the set (space) of potential alternatives, X_0 is the set of feasible alternatives, $f : \mathcal{X} \to \mathcal{R}^k$ is the criteria map in which $f = (f_1, ..., f_k)$, and $f_i : \mathcal{X} \to \mathcal{R}$ are *criteria functions*, $i = 1, ..., k,\ k \geq 2$. Without loss of generality we assume that all criteria are of "better if more" type.

An alternative x for which $f(x)$ is the most preferred vector of criteria values, is the *most preferred* alternative.

Without any ambiguity alternatives are represented by their criteria values. With this in mind we deal mainly with elements $f(x)$ of set $f(X_0)$ and for the sake of simplicity we denote

$$y = f(x), \text{ and } Z = f(X_0).$$

Clearly, $Z \subseteq \mathcal{R}^k$. Throughout the paper we assume that Z is closed. Elements of set Z are called *outcomes* and space $\mathcal{R}^k$ is called the *outcome space*. Under this convention, for a given feasible alternative x, y_i is the value of the $i-$th component of outcome $y = f(x)$. Thus, y_i is the value of $i-$th criterion for alternative x.

Referring to notation x, X_0, $f(x)$, $f(X_0)$ is needed only when giving examples of MCDM problems with implicitly (i.e. in the form of constraints) defined feasible alternatives.

Element $\hat{y}$ of $\mathcal{R}^k$, called *utopian,* is calculated as

$$\hat{y}_i = max_{y \in Z}\ y_i,\ i = 1, ..., k.$$

Throughout this paper it is assumed that all these maxima exist. Element $\hat{y}$ need not represent any alternative.

Below we make use of a selected element of outcome space $\mathcal{R}^k$, denoted y^*, defined as

$$y_i^* = \hat{y}_i + \epsilon,\ i = 1, ..., k,$$

where ϵ is any positive number.

In this paper we refer to standard definitions of efficiency and weak efficiency (in the sense of Pareto). We say that outcome $\bar{y} \in Z$ is *efficient* if $y_i \geq \bar{y}_i,\ i = 1, \ldots, k,\ y \in Z$, implies $y = \bar{y}$ and is *weakly efficient* if there is no $y \in Z$ such that $y_i > \bar{y}_i,\ i = 1, \ldots, k$.

3. Capturing DM Preferences with Scalarizing Functions

An outcome which minimizes (or, depending on function, maximizes) a scalarizing function over Z is efficient (at worst - weakly efficient). From a variety of scalarizing functions proposed we focus here only on those which have favorable computational properties (cf. Wierzbicki 1986) and are widely used both in methodological MCDM considerations (Steuer 1986, Wierzbicki 1986,1999, Kaliszewski 1994,2004,2006, Miettinnen 1999) and MCDM applications (Makowski et al. 1996, Gal et al. 1999, Wierzbicki et al. 2000).

In general, a scalarizing function has two parameters: a vector of *weights* λ and a *reference point* y (element of the outcome space, cf Section 2, i.e.

$$f_{\lambda,y}(\cdot)\,. \tag{2}$$

An exception is the *linear scalarizing function*, which has only one parameter, namely vector λ. This function will be discussed separately.

Depending on the role of λ and y, scalarizing functions define two major classes of MCDM methods: the weight method class and the reference point method class.

In the weight method class DM's preferences are captured by the mechanism of weights. A scalarizing function with the reference point fixed and the vector of weights selected by the DM, when minimized over Z yields an efficient (at worst - weakly efficient) outcome. For presentations of methods of this class see e.g. Steuer (1986), Miettinen (1999), Kaliszewski (2004,2006).

In the reference point method class DM's preferences are captured by the mechanism of reference points. A scalarizing function with the vector of weights fixed and the reference point selected by the DM, when minimized over Z, yields an efficient (at worst - weakly efficient) outcome. For presentations of methods of this class see Wierzbicki (1980,1986,1999), also e.g. Miettinen (1999), Kaliszewski (2004,2006).

In Kaliszewski (2004,2006) it is shown that in fact those two classes of methods in technical terms are equivalent to selecting (a priori or at each iteration of a decision making process, depending on the manner of solving the problem) a point of the outcome space and a direction in this space.

We recall now definitions of scalarizing functions used in the sequel.

Let λ be a vector of weights, $\lambda = (\lambda_1, ..., \lambda_k),\ \lambda_j > 0,\ j = 1, ..., k$.

The linear (weighted) scalarizing function is defined as

$$\sum_{j=1}^{k} \lambda_j y_j, \tag{3}$$

where y are elements of the outcome space. To derive efficient outcomes such functions have to be maximized.

Let λ be vectors of weights as above. The (modified weighted) *Tchebycheff scalarizing function* is defined as

$$\max_j \lambda_j((y_j^{ref} - y_j) + \rho e^k(y^{ref} - y)), \tag{4}$$

where y^{ref} is a reference point, e^k is k-dimensional vector $(1, 1, ..., 1)$, $0 \leq \rho < +\infty$. To derive efficient outcomes such functions have to be minimized. We skip discussing the other possible form of the Tchebycheff-type scalarizing function (cf. Steuer 1986, Wierzbicki 1986,1999, Kaliszewski 1994,2004,2006, Miettinnen 1999) for the differences between those two forms pertain only to technicalities.

The linear scalarizing function and the Tchebycheff scalarizing function are the forms of scalarizing functions most often exploited.

In the case of the linear scalarizing function the DM has to specify vector λ.

In the case of the Tchebycheff scalarizing function both vector λ and reference point y^{ref} are to be specified. Depending on the MCDM method class, the DM specifies either vector λ (as in the weight method class) or reference point y^{ref} (as in the reference point method class). The complementing element is specified according to rules of the specific method applied.

It is quite natural and quite easy for the DM to provide reference points, and this makes the reference point methods so versatile. In the weight method class, if the Tchebycheff scalarization function is used, reference point is, as a rule, fixed to y^* (cf. Section 2) reducing (4) to

$$\max_j \lambda_j((y_j^* - y_j) + \rho e^k(y^* - y)). \qquad (5)$$

However, it is much less natural for the DM to provide vectors λ. There are two reasons for that. First, the DM may not have a priori any clear view of the relative importance of criteria prior to seeing consequences (outcomes), resulting from various weight patterns represented by vectors λ. Second, there is no general method to establish relations between weights and outcomes other than "*pick* (weights) *and derive* (outcomes)".

The relative importance of criteria can be elicited from the DM (a priori or at each iteration of a decision making process) by asking him to specify two reference points, as in the extended form of the reference point methods: the reservation point and the aspiration point (cf. Wierzbicki 1999). The line segment between these two points defines the relative importance of the criteria, as perceived by the DM. In other words, this line segment defines proportions between values of criteria the DM wants to be kept when searching for a preferred alternative. Thus, this line segment defines a consensus line, as defined verbally in Section 1. Given a consensus direction, an outcome more preferred than the reference point should be searched for on the consensus line. (cf. Figure 1). Any outcome off the consensus line clearly does not satisfy the DM preference expressed in the form of consensus direction.

The main deficiency of scalarizing functions is that in general they may provide for efficient outcomes which do not reflect the relative importance of criteria as specified by the DM (cf. Figure 2). For the linear scalarizing function as well as for the Tchebycheff scalarizing function the difference between the pattern of criteria values expected and the pattern represented by the derived outcome can be significant. This results from the fact that search for efficient outcomes is done by maximizing or minimizing a function over the outcome set, hence all elements on an isogram of the function are treated as "preference equivalent".

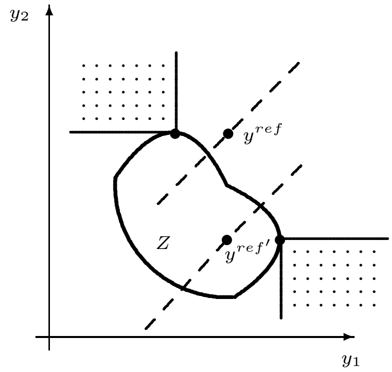

Figure 1: Searching along consensus lines for more preferred outcomes than given reference points.

The linear function, by virtue of its nature, totally ignores the distance from outcomes to the consensus line. On the other hand, the Tchebycheff scalarizing function is a proxy for consensus lines in the following sense. Suppose two elements y and y^{ref} of a consensus line such that either $y_i > y_i^{ref},\ i = 1, ..., k$, or $y_i < y_i^{ref},\ i = 1, ..., k$, are given. If

$$\lambda_i = |(y_i^{ref} - y_i) + \rho e^k (y^{ref} - y)|^{-1},\ i = 1, ..., k, \tag{6}$$

then all the apexes of isograms of the Tchebycheff scalarizing function (which represent displaced cones) lie on this consensus line (cf. Kaliszewski 2006).

If Z is convex, $\rho = 0$, λ is defined by (6), and the consensus line intercepts Z, the apex of the Tchebycheff function isogram which from all isograms intercepting Z has the minimal value, is an efficient outcome. This observation extends to outcome sets which are R_+^k-convex (Z is R_+^k-convex if $Z - R_+^k$ is convex, where R_+^k is the nonnegative orthant of $\mathcal{R}^k$) (Figure 3).

With $0 < \rho < +\infty$ the Tchebycheff scalarizing function may derive outcomes off the consensus line but still closer (in the sense e.g. of the Euclidean norm) than outcomes derived by the linear scalarizing function. This observation relates to the notion of equity (cf. e.g. Kostreva et al. 2004, Ogryczak 2007).

The DM provided with efficient outcomes for subsequent evaluations and preference expressing should know if an efficient outcome is found on or off the consensus line. If it is found off the line, a certain measure of such a "defect" should be available to him. However, current use of scalarizing functions in the framework of interactive MCDM ignores this information.

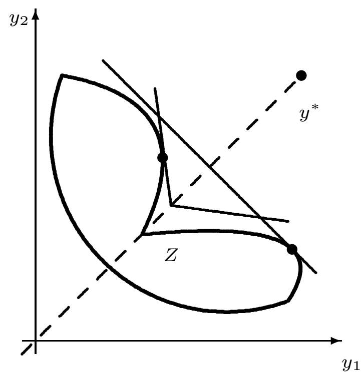

Figure 2: Deriving efficient outcomes with the linear function and the Tchebycheff function ($\rho \neq 0,\ \lambda = e^k$).

4. Capturing DM Preferences with Consensus Directions

To capture DM preferences more accurately (i.e. to keep more closely proportions between values of criteria, as specified by the DM), we postulate searching outcomes along consensus lines (cf. Figure 1).

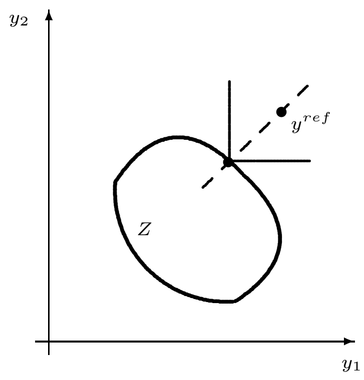

Figure 3: If Z is R^k_+-convex, $\rho = 0$, λ is defined by (6), and the consensus line intercepts Z, the Tchebycheff function and searching along the consensus line derive the same efficient outcomes.

A consensus line is formally defined as

$$y = y^{ref} + \tau t, \tag{7}$$

where y^{ref} is a reference point and τ is a *consensus direction*, $t \in \mathcal{R}$.

As assumed in Section 1, consensus directions are non-negative, i.e. they have all elements non-negative. This assumption puts some limitations on selection of reference points, as illustrated in Figure 4.

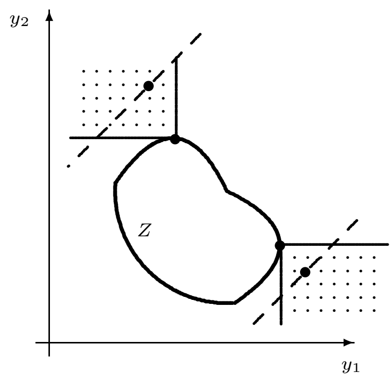

Figure 4: No consensus line starting from a reference point selected from the dotted regions meets an efficient outcome.

The DM can define consensus directions in two ways. The first way is to define consensus direction τ directly by specifying a vector of positive components. The second way is to define a consensus direction implicitly by specifying a pair of reference points, a *reservation point* and an *aspiration point*, say y' and y'', such that $y' - y''$ or $y'' - y'$ defines consensus direction τ.

In general, the postulate that outcomes should be searched for along consensus lines should be handled with care. First, it might happen than no outcome on the consensus line is efficient. Second, if Z is finite, it might happen that the consensus line contains no outcome at all (Figure 5).[3] To heal this the search for outcomes should be made not only on consensus lines but also in some neighborhood of them.

We define a neighborhood for each point of the consensus line with the help of functions. Given a function, for each element y of the consensus line its neighborhood is composed of points which function value is equal to $f(y)$. For example, the Tchebycheff scalarizing function (4) defines for $0 \leq \rho < +\infty$ neighborhoods, which are displaced cones (Figure 2).

To simplify presentation we assume that each time the DM specifies a new consensus direction τ the problem considered is scaled in the following way: $y'_i = (\tau_i)^{-1} y_i,\ i = 1, ..., k$. In consequence, after scaling consensus directions and consensus lines take forms

$$\tau'_i = (\tau_i)^{-1} \tau_i = 1,\ i = 1, ..., k,$$

[3]This may happen also when Z is infinite. To avoid this some limitations on the choice of λ have to be imposed, as shown in Kaliszewski (2006).

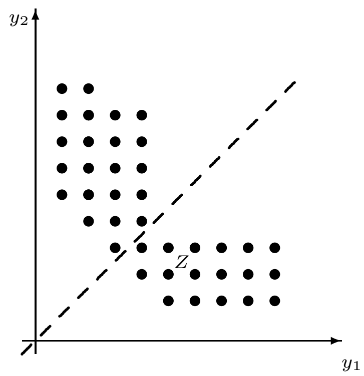

Figure 5: The line search does not always derive an outcome.

$$y'_i = (\tau_i)^{-1} y_i^{ref} + t(\tau_i)^{-1}\tau_i = (y')_i^{ref} + t,\ i = 1, ..., k, \tag{8}$$

or for short

$$\tau' = e^k,$$
$$y' = (y')^{ref} + t\, e^k.$$

Observe that after scaling the Tchebycheff scalarizing function (4) which all apexes lie on the consensus line reduces to

$$\max_j((y_j^{ref} - y_j) + \rho e^k (y^{ref} - y)). \tag{9}$$

On the other hand, for $\rho \to +\infty$ minimization of function (9) is equivalent to maximization of the linear function

$$\sum_{j=1}^{k} y_j, \tag{10}$$

For proofs see Appendix A.

For the sake of notational simplicity from now on we drop apostrophes indicating that the problem has been scaled.

To define neighborhoods of elements of consensus lines we propose to use function (9) with negative values of ρ, $-\frac{1}{k} \leq \rho < 0$. The rationale for that is the fact that for $\rho = -\frac{1}{k}$ isograms of the function (9) reduce to the consensus line (for proof see Appendix B). On the other hand, for $\rho = 0$ function (9) becomes the Tchebycheff scalarizing function, which guarantees efficiency of outcomes derived with it. Values of $\rho < -\frac{1}{k}$ are meaningless in the decision making context.

With function (9) and $\frac{1}{k} \leq \rho \leq 0$, outcomes are derived either on the consensus or are of a smaller or at most of equal distance to the consensus line than those derived by the Tchebycheff scalarizing function, i.e. function (9) with $0 \leq \rho < +\infty$ (Figure 6).

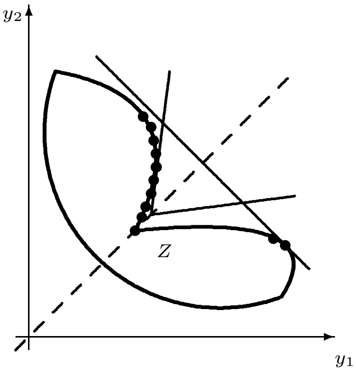

Figure 6: Outcomes derived with scalarizing functions (9) and (10) as ρ varies from $-\frac{1}{k}$ to $+\infty$.

5. Numerical Example

As an example we consider an instance of the Markowitz mean-variance portfolio selection problem. In this problem the most preferred portfolio is selected from available stocks to maximize portfolio expected return (as a measure of gain) and minimize portfolio variance (as a measure of risk). The problem comprises one linear and one quadratic criterion over a continuum of possible combinations of available stocks. To be consistent with our assumption that all criteria are "the more the better", we maximize the negative of variance.

Selected stocks are to consume all the available capital normalized to a unit, so stock participation in the portfolio is represented by fractions (variables x) which all sum to one. We consider the case where borrowing (short sale) of stocks is not allowed, so stock fraction in the portfolio has to be nonnegative. In addition, we consider the case where some stock, if selected for the portfolio, can be only at or above a certain minimal level.

The formulation of the problem is as follows. Let $I = \{1, ..., k\}$.

$\max -\sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} x_i x_j$ (maximize negative of portfolio variance)

$\max \sum_{i=1}^{n} e_i x_i$ (maximize portfolio expected return)

$\sum_{i=1}^{n} x_i = 1,$ ("all the capital" constraint)

$m_i \delta_i \le x_i \le \delta_i\,,$ $\delta_i = 0$ or $1,\ i \in J \subseteq I\,,$ ("no small trades" and nonnegativity constraints for the selected stock)

$x_i \ge 0,\ i \in I \setminus J\,,$ (nonnegativity constraints for all the remaining stock) where β_{ij} denotes the covariance matrix coefficient for stock i and stock j, e_i and m_i denote the expected return and the minimal level for stock i, respectively, $I = \{1, ..., k\}$. Binary variables cause the problem to be nonconvex.

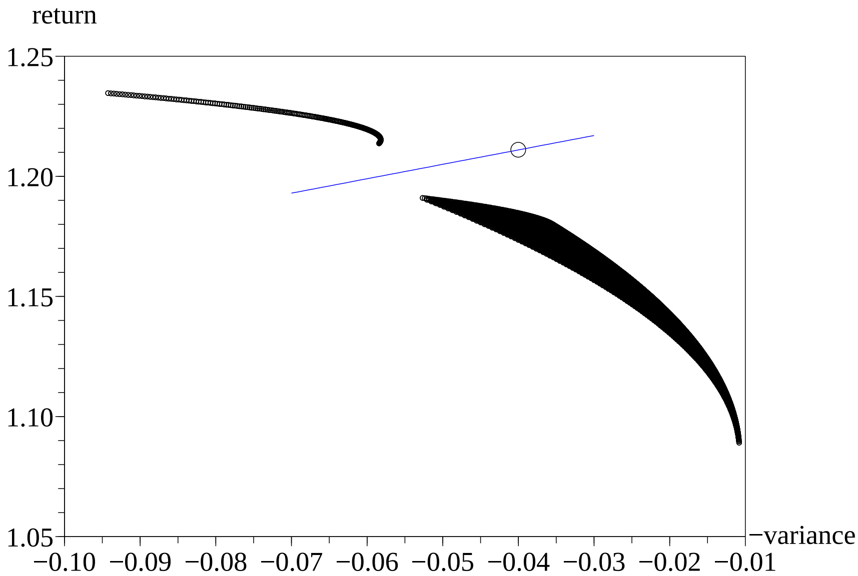

Figure 7: Illustration to numerical example of Section 5.

Data is taken from the well-known "three-stock" example by Markowitz and we amend it with arbitrarily selected numbers for m_i. In the Markowitz example there are three stocks denoted ATT, GMC, USX, characterized by the following covariance matrix and the expected returns over one investment period:

	ATT	GMC	USX	
ATT	0.01080754	0.01240721	0.01307513	
GMC	0.01240721	0.05839170	0.05542639	covariance matrix
USX	0.01307513	0.05542639	0.09422681	
	0.0890833	0.213667	0.234583	expected returns
	0.3	none	none	minimal levels

The example is small but illustrative, for no matter what the number of stocks is considered, decisions are taken in the two-dimensional, return-variance space.

Let $y_{-v} = -\sum_{i=1}^{n}\sum_{j=1}^{n}\beta_{ij}x_ix_j$ and $y_e = \sum_{i=1}^{n} e_ix_i$.

The outcome set of the problem is represented in Figure 7. Because of binary variables this set is disjoint.

We assume that at a certain stage of the search for his most preferred portfolio the DM selected reference point $(y_{-v}, y_e) = (-0.04, 1.211)$ and consensus direction $\tau = (1.0, 0.6)$. The resulting consensus line does not intersect the outcome set (Figure 7).

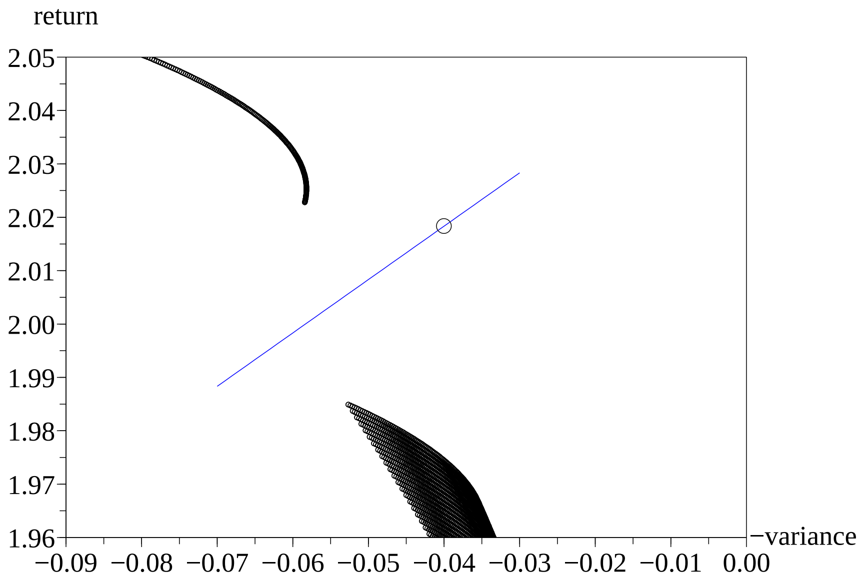

Figure 8: Illustration to numerical example of Section 5, cont. 1.

Now we scale the problem with the consensus direction components, as proposed in Section 4. The scaled outcome set, the reference point, and the consensus line take the form as shown in Figure 8 (this figure represents a fragment of the outcome set around the consensus line).

For a number of values of ρ such that $-\frac{1}{k} < \rho < +\infty$ we derive outcomes for the scaled problem using function (9). The results are presented in Figure 9 and Table 1. It is easy to relate outcomes in Table 1 to outcomes in Figure 9 since outcomes in Table 1 are ordered from 1 to 10 with decreasing values of expected return (the second component). For each outcome derived its (Euclidean) distance to the consensus line is calculated. Column 3 and 4 present data for the scaled problem, column 5 and 6 for the original problem. Outcome #8 and #9 are nonefficient and the remaining 8 outcomes are efficient. The outcome with the smallest distance to the consensus line is (efficient) outcome #10. But clearly with other consensus line a nonefficient outcome (e.g.outcome #9) can become the one with the smallest distance to that line.

Reference point $(-y_{-v}, y_e) = (-0.04, 1.211)$ and outcome#10 derived with $\rho = -0.47$ define consensus direction $(1.0, 1.6)$, whereas the same reference point and outcome #7 derived with $\rho = 0$ define consensus direction $(1.0, -0.23)$. The realized consensus represented by either of these two outcomes is quite apart of what the DM postulated in the form of consensus direction (he postulated consensus direction (1,1) in the scaled problem) and then expected in the form of proportions of outcome component values.

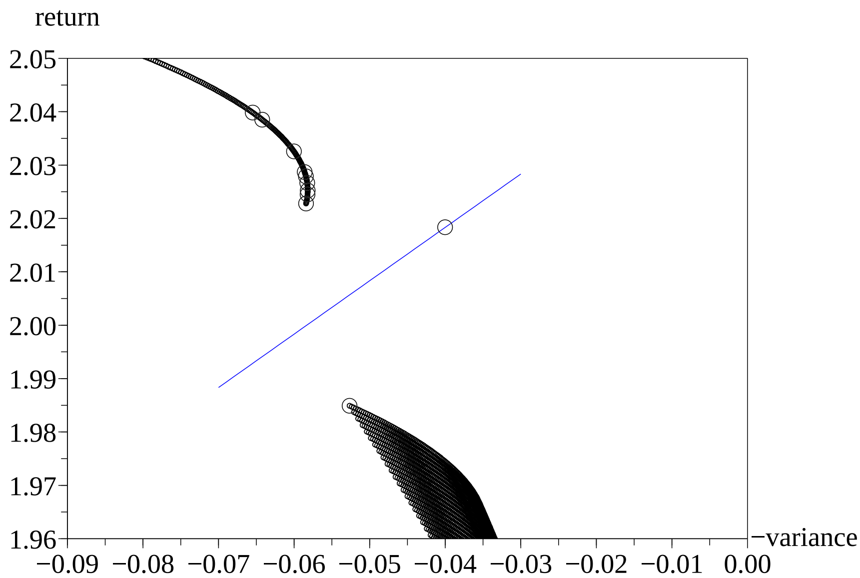

Figure 9: Illustration to numerical example of Section 5, cont. 2.

6. Discussion

Once again the "beauty" of convex problems has manifested. In this case searching for outcomes along consensus lines, under some restrictions on selections of reference points and consensus directions specified in Section 3, derives efficient outcomes. Therefore, searching for outcomes along consensus lines is equivalent to searching for outcomes with Tchebycheff scalarizing function with $\rho = 0$.

However, even if a problem in its very nature is convex, side constraints, so common in practical applications of decision problems, can easily relocate the problem from the convex to the nonconvex class. The example of Section 5 provides a convincing illustration of that.

As seen above, when dealing with nonconvex problems supporting decision making calls for the utmost care. A schematic application of existing tools could be deceptive to the DM. In the example of Section 5 the Tchebycheff function with nonnegative values of ρ was able to derive efficient outcomes but none of them was the efficient outcome closest to the consensus line. Thus, the compromise of component (criteria) values offered by such outcomes do not always follow closely the consensus specified (and hence expected) by the DM.

In the case where the outcome closest to the consensus line is efficient there is no doubt that this and no other outcome should be derived and presented to the DM for evaluation. The situation is less clear if the outcome closest to the consensus line is nonefficient. In that case, in virtue of efficiency, there exists another outcome which offers more attractive values of criteria (in other words it dominates the outcome closest to the consensus

Table 1: Outcomes derived for various ρ in the numerical example of Section 5.

		Scaled problem		Original problem	
ρ	#	y	Dist.	y	Dist.
∞	1	$(-0.06545, 2.03980)$	0.03318	(-0.06545, 1.22388)	0.02711
10.00	2	$(-0.06419, 2.03848)$	0.03135	(-0.06419, 1.22309)	0.02565
1.00	3	$(-0.06000, 2.03253)$	0.02418	(-0.06000, 1.21952)	0.02016
0.30	4	$(-0.05858, 2.02866)$	0.02044	(-0.05858, 1.21720)	0.01752
0.20	5	$(-0.05841, 2.02786)$	0.01976	(-0.05841, 1,21672)	0.01706
0.10	6	$(-0.05824, 2.02661)$	0.01875	(-0.05824, 1.21597)	0.01641
0.00	7	$(-0.05818, 2.02520)$	0.01771	(-0.05818, 1.21519)	0.01577
-0.05	8	$(-0.05820, 2.02446)$	0.01721	(-0.05820, 1.21469)	0.01547
-0.10	9	$(-0.05840, 2.02279)$	0.01615	(-0.05840, 1.21367)	0.01489
-0.20	9	$(-0.05840, 2.02279)$	0.01615	(-0.05840, 1.21367)	0.01489
-0.30	9	$(-0.05840, 2.02279)$	0.01615	(-0.05840, 1.21367)	0.01489
-0.40	9	$(-0.05840, 2.02279)$	0.01615	(-0.05840, 1.21367)	0.01489
-0.45	9	$(-0.05840, 2.02279)$	0.01615	(-0.05840, 1.21367)	0.01489
-0.46	9	$(-0.05840, 2.02279)$	0.01615	(-0.05840, 1.21367)	0.01489
-0.47	10	$(-0.05263, 1.98489)$	0.01471	(-0.05263, 1.19093)	0.00525
-0.48	10	$(-0.05263, 1.98489)$	0.01471	(-0.05263, 1.19093)	0.00525

line). Being more attractive with respect to efficiency this outcome is less attractive with respect to distance to the consensus line. So which of those two is more preferred?[4]

The model (1) does not provide information to resolve this question. Thus, consistently with the interactive MCDM approach, it should be left to the DM to decide. For this aim we should present to the DM a set of outcomes ordered with increasing distance to the consensus line, starting from the outcome closest to the consensus line to the first efficient outcome. In the example of Section 5 such set is a singleton, namely outcome #10.

Presenting at each iteration a range of outcomes to the DM for evaluation does not add to the clarity and simplicity of interactive decision making but seems to be indispensable to avoid loss of available information.

Deriving outcomes with function (9) and negative values of ρ we are in position to extend search for the most preferred outcomes (alternatives) beyond limits set by the current use of scalarizing functions.

The idea of line search has been already exploited in a number of MCDM papers (cf. e.g. Korhonen 1988, Jaszkiewicz, Słowiński 1995) but in all of them authors stick to the standard use of scalarizing functions. As shown in this paper, line search can bring novel elements to MCDM only for problems models which escape the convexity assumption.

[4] Even if one prefers the dominated (i.e. nonefficient) outcome, it is not that he is at odds with the efficiency dogma. Indeed, this outcome is efficient when considered in k original criteria and "the distance to the compromise line" criterion, since it extremizes the latter.

It has been shown in Kaliszewski (2004,2006) that in technical terms line search can be interpreted as either a member of the reference point method class or of the weighting method class.

7. Concluding Remarks

A fair consensus among conflicting criteria is at the heart of multiple criteria decision making. Nowadays interactive MCDM methods gain popularity since they do not require the DMs to be involved in the process of complete preference revealing and modeling, in which he does not willingly participate. Instead, the DM evaluates a series of outcomes and his partial preferences are used to structure the search for the most preferred outcome. Proper process structuring depends on the appropriateness of the information which can be derived form problem model (1).

It is therefore important that DM's partial preferences are properly reflected in decision processes in the form of outcomes presented to him for evaluation. In this paper we have elaborated on this.

Proper reflection of DM's partial preferences in decision processes is of particular importance in problems where there is no or little room for an extensive search over the outcome set, as e.g in public procurement problems, or in negotiations involving conflicting and noncooperating parties.

Appendix A

By formula (6), after scaling, consensus direction $\tau' = e^k$ is related to vector λ' which ensures that all the apexes of the Tchebycheff scalarizing function with λ' lie on the consensus line, as follows

$$\lambda'_i = |((y'_i)^{ref} - y'_i) + \rho e^k((y')^{ref} - y')|^{-1},\ i = 1, ..., k,$$

where (by formula (8)) $y' = (y')^{ref} + t\,e^k\ ,\ t \in \mathcal{R},\ y'_i \neq (y')^{ref}_i,\ i = 1, ..., k.$

Hence,

$$\begin{aligned}\lambda'_i &= |((y'_i)^{ref} - (y'_i)^{ref} - t) + \rho e^k((y')^{ref} - (y')^{ref} - t\,e^k)|^{-1} \\ &= |-(k+1)t| = (k+1)t,\ t \in \mathcal{R},\ i = 1, ..., k.\end{aligned}$$

Thus, all elements of vector λ are equal and therefore, for convenience and without loss of generality, can be replaced by 1.

Since multiplication of the objective function by a constant does not change optimal solutions, function (9) can be reduced to the equivalent form

$$\max_j\left(\frac{y^{ref}_j - y_j}{\rho} + e^k(y^{ref} - y)\right),$$

which for $\rho \to +\infty$ tends to

$$\max_j(e^k(y^{ref} - y)) = e^k(y^{ref} - y) = \sum_{i=1}^{k}(y^{ref}_i - y_i).$$

It is clear that minimization of the last term is equivalent to maximization of

$$\sum_{i=1}^{k} y_i \,.$$

Appendix B

Lemma *Suppose that the problem considered has been scaled as shown in Section 4. For $\rho = -\frac{1}{k}$ isograms of function (9) reduce to the consensus line.*

Proof. After scaling the consensus line has the form $y = y^{ref} + t\, e^k$. Suppose $\bar{y} \in \mathcal{R}^k$ is an element of the consensus line. The isogram of function (9) with apex at $\bar{y}$ is defined as

$$\max_j (y_j^{ref} - \bar{y}_j) - \frac{1}{k}\sum_{i=1}^{k}(y_i^{ref} - \bar{y}_i)$$
$$= \max_j (y_j^{ref} - y_j) - \frac{1}{k}\sum_{i=1}^{k}(y_i^{ref} - y_i). \tag{11}$$

Observe first that the following holds

$$(y_j^{ref} - \bar{y}_j) - \frac{1}{k}\sum_{i=1}^{k}(y_i^{ref} - \bar{y}_i) = 0 \text{ for each } j,\ j = 1, ..., k.$$

Indeed, for $\bar{y}_j = y_j^{ref} + t,\ t \in \mathcal{R},\ j = 1, ..., k$, we get

$$(y_j^{ref} - \bar{y}_j) - \frac{1}{k}\sum_{i=1}^{k}(y_i^{ref} - \bar{y}_i)$$
$$= (y_j^{ref} - y_j^{ref} - t) - \frac{1}{k}\sum_{i=1}^{k}(y_i^{ref} - y_i^{ref} - t)$$
$$= -t + \frac{1}{k}\sum_{i=1}^{k} t = 0,\ j = 1, ..., k.$$

Hence, (11) is equivalent to

$$0 = \max_j (y_j^{ref} - y_j) - \frac{1}{k}\sum_{i=1}^{k}(y_i^{ref} - y_i).$$

Observe that for any element of $\mathcal{R}^k$ the following relation holds

$$\max_j (y_j^{ref} - y_j) \geq \frac{1}{k}\sum_{i=1}^{k}(y_i^{ref} - y_i).$$

Indeed, the largest element in a set of numbers is never smaller than the average of numbers. Moreover, the largest number is equal to the average only if all numbers are equal. Thus

$$0 = \max_j(y_j^{ref} - y_j) - \frac{1}{k}\sum_{i=1}^{k}(y_i^{ref} - y_i)$$

holds only if

$$y_1^{ref} - y_1 = ... = y_k^{ref} - y_k = -t,\ t \in \mathcal{R},$$

or equivalently,

$$y = y^{ref} + e^k t,\ t \in \mathcal{R},$$

which is the consensus line of the scaled problem.

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